# APPROXIMATION OF LIPSCHITZ FUNCTION BY (E,q) (C,1) SUMMABILITY METHOD

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## Abstract

In the paper a new theorem on the degree of approximation of Lipschitz function on by (c,1) means is established

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## Introduction

First time in 1991 Lal [ 2] has been discussed the degree of approximation by (c,1) (E,1) product summability means. The degree of approximation of function belonging to generalized Lipschitz class by Nörlund means has been determined by several researchers of modern analysis. But nothing seems to have been done so for in the direction of study of degree of approximation of Lipα function by product summability means of the form (E,q) (c,1). In present paper, the degree of approximation of Lipschitz function by (E,q) (c,1) means of its Fourier series has been determined.

## Definition and Notations

Let f(t) be periodic with period $2\pi$ and intergrable in the sense of Lebesgue. The Fourier series of f(t) is given by

$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \qquad (2.1)$$

A function f∈ Lip∝ if

$$f(x+t) - f(x) = O(|t|^{\propto}) \ \ for\, 0 < \propto \leq 1$$

The degree of approximation of a function f:R→R by a trigonometric polynomial $t_n$ order n is defined by, zygmund (1968,1,p-114.).

$$\left\| t_n - f \right\|_\infty = \sup \left\{ \left| t_n(x) - f(x) \right| : x \in R \right\}$$

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with the sequence of partial sum $s_n$.

If

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} s_k \to s \quad as \quad n \to \infty$$

Then The Series $\sum_{n=0}^{\infty} u_n$ is said to Summable to s by $(C,1)$ method.

For q>0, the Euler mean is given by Hardy

$$E_n^q = (1+q)^{-n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} s_k$$

and the method is denoted as $(E,q)$

$$If \ E_n^q \to s, n \to \infty$$

Then $\sum_{n=0}^{\infty} u_n$ is said to summable to s by $(E,q)$ method.

$(E,q)$ transformation of $\sigma_n$ i.e $(E,q)(c,1)$ transform of $\left\{ s_n \right\}$ is defined by

$$E_n^{q,c_1} = \frac{1}{(1+q)^n} \sum_{k=o}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} \sigma_k$$

$$\text{if} \quad E_n^{q,c_1} \to s \quad as \quad n \to \infty$$

Then, we say that $\sum_{n=0}^{\infty} u_n$ is said to be summable to s by $(E,q)c_1$ method.

It is denoted by $\sum_{n=0}^{\infty} u_n = s\left((E,q)c_1\right)$

We shall use the following notation

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

$$E_n^{q,c_1} = \frac{1}{2\pi(1+q)^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} \frac{\sin^2 (k+1) t/2}{(k+1)\sin^2 t/2}$$

## 3. Main Theorem

The degree of approximation of $f \in Lip\propto$ has been determined in the following form:

**Theorem**: If $f: R \rightarrow R$ is $2\pi$- periodic and Legesgue integrable on $[-\pi, \pi]$ and $f \in Lip \propto$ then the degree of approximation of f by (E,q)(c,1) product summability means of its Fourier series is given by

$$\left\| E_n^{q,c_1}(x) - f(x) \right\|_\infty = \begin{Bmatrix} O\dfrac{1}{(n+1)^\propto}, 0 < \propto < 1 \\ O\dfrac{\log(n+1)\pi e}{(n+1)}, \propto = 1 \end{Bmatrix}$$

where $E_n^{q,c_1}(x) = \dfrac{1}{(1+q)^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} \left( \dfrac{1}{k+1} \sum_{\upsilon=0}^{k} s_\upsilon \right)$

is $(E,q)(c,1)$ means of fourier series (2.1).

## 4. Lemmas

For the proof of our theorem the following Lemmas are required.

**Lemma 4.1:-** For $0 \le t \le \frac{1}{n+1}$

**Proof:-**

$$E_q^{n,c_1}(t) = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{n-k}}{k+1} \frac{\sin^2 (k+1) t/2}{\sin^2 t/2}$$

$$\left| E_q^{n,c_1}(t) \right| = \left| \frac{1}{(1+q)^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{n-k}}{k+1} \frac{\sin^2 (k+1) t/2}{\sin^2 t/2} \right|$$

$$\le \frac{1}{(1+q)^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{n-k}}{k+1} \frac{(k+1) \sin^2 t/2}{\sin^2 t/2}$$

$$\le \frac{1}{(1+q)^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} (k+1)$$

$$= O(n+1) \frac{1}{(1+q)^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} = O(n+1)$$

**Lemma 4.2:-** For $\frac{1}{(n+1)} \le t \le \pi, \frac{1}{\sin\frac{t}{2}} \le, \frac{\pi}{t}$ by Jondan's Lemma.

$$\left|E_q^{n,c_1}(t)\right| = \left|\frac{1}{(1+q)^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{q^{n-k}}{(k+1)}\frac{\sin^2 (k+1)\,t/2}{\sin^2 t/2}\right|$$

$$= \frac{1}{(1+q)^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{q^{n-k}}{(k+1)}\frac{\left|\sin^2 (k+1)\,t/2\right|}{\sin^2 t/2}$$

$$\le \frac{1}{(1+q)^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{q^{n-k}}{(k+1)}\left(\frac{\pi}{t}\right)^2$$

$$= \frac{\pi^2}{t^2}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{q^{n-k}}{(k+1)}$$

$$O\left(\frac{1}{(n+1)t^2}\right)$$

## 5. Proof of the main theorem

Let $s_n(x)$ denote n[th] partial sum of series (2.1) at t=x then following Tichmarsh (p.142 &143).

$$\sigma_n = \sum_{k=0}^{n}\frac{s_k}{n+1} = (C,1) \text{ mean of } s_n(x).$$

$$= f(x) + \frac{1}{2(n+1)\pi}\int_0^{\pi}\frac{\sin^2\frac{(n+1)t}{2}}{\sin^2 t/2}\left\{f(x+t)+f(x-t)-2f(x)\right\}\phi(t)dt$$

$$\sigma_n - f(x) = \frac{1}{2(n+1)\pi}\int_0^{\pi}\frac{\sin^2\frac{(n+1)t}{2}}{\sin^2 t/2}\,\phi(t)dt$$

Now, $\frac{1}{(1+q)^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} q^{n-k}\left\{\sigma_k - f(x)\right\}$

$$= \frac{1}{2\pi}\int_0^{\pi}\frac{1}{(1+q)^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{q^{n-k}\sin^2\frac{(k+1)t}{2}}{(k+1)\sin^2 t/2}\phi(t)dt$$

$$= \frac{1}{2\pi}\int_0^{\pi}\frac{1}{(1+q)^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{q^{n-k}}{(k+1)}\frac{\sin^2\frac{(k+1)t}{2}}{\sin^2 t/2}\phi(t)dt$$

$$= \frac{1}{2\pi}\int_0^{\pi}\mathrm{E}_n^{q,c1}(t)\phi(t)dt$$

Let us consider $I_1$ at first

$$\left|I_1\right| = \left|\frac{1}{2\pi}\right|\int_0^{1/(n+1)}\left|\mathrm{E}_n^{q,c1}(t)\right|\left|\phi(t)\right| dt$$

$$= O(n+1)\int_0^{1/(n+1)}\left|\phi(t)\right| dt \qquad \text{by lemma (4.1)}$$

$$= O(n+1)\int_0^{1/(n+1)} o(t^{\propto})dt \qquad \because \phi(t) \in Lip \propto$$

$$= O\frac{(n+1)}{(\propto+1)}\left(\frac{1}{(n+1)^{\propto+1}} - 0\right)$$

$$= \frac{1}{(\propto +1)} \frac{1}{(n+1)^{\propto}}$$

Let us consider $I_2$.

$$I_2 = \int_{\frac{1}{n+1}}^{\pi} E_n^{q,c1}(t)\phi(t)dt$$

$$= \int_{\frac{1}{n+1}}^{\pi} O\left(\frac{1}{(n+1)t^2}\right)\phi(t)dt \qquad \text{by lemma(4.2)}$$

$$= \int_{\frac{1}{n+1}}^{\pi} O\left(\frac{1}{n+1}\right)t^{\propto-2}dt$$

$$= O\left(\frac{1}{n+1}\right)\int_{\frac{1}{n+1}}^{\pi} t^{\propto-2}dt$$

$$= O\left(\frac{1}{n+1}\right)\left[\frac{t^{\propto-1}}{\propto-1}\right]_{n+1}^{\pi}$$

$$= \begin{cases} O\left(\frac{1}{n+1}\right)\left[\frac{\left(\frac{1}{n+1}\right)^{\propto-1}}{\propto -1} - \frac{\pi^{\propto-1}}{(\propto -1)}\right], 0 <\propto< 1 \\ O\left(\frac{1}{n+1}\right)\log(n+1)\pi, \qquad \propto= 1 \end{cases}$$

$$= \begin{cases} O\left(\frac{1}{n+1}\right)\frac{\left(\frac{1}{n+1}\right)^{\propto-1}}{\propto-1} - \pi^{\propto-1}, 0<\propto<1 \\ O\left(\frac{1}{n+1}\right)\log(n+1)\pi, \quad \propto=1 \end{cases}$$

$$\leq \begin{cases} \left(\frac{1}{1-\propto}\right)\left(\frac{1}{(n+1)^{\propto}}\right), 0<\propto<1 \\ \frac{\log(n+1)^{\pi}}{(n+1)}, \propto=1 \end{cases}$$

Now,

$$I_1 + I_2 = \begin{cases} \frac{1}{(\propto+1)}\frac{1}{(n+1)^{\propto}} + \frac{1}{(1-\propto)}\frac{1}{(n+1)^{\propto}}, 0<\propto<1 \\ \frac{1}{2(n+1)} + \frac{\log(n+1)\pi}{(n+1)}, \propto=1 \end{cases}$$

$$= \begin{cases} O\left(\frac{1}{(n+1)^{\propto}}\right), 0<\propto<1 \\ O\left(\frac{O(\log(n+1)\pi e)}{n+1}\right), \propto=1 \end{cases}$$

This is complete the proof of the main theorem.

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